ON THE QUESTION OF THE DYADIC OPERATION OF DYADIC GREEN'S FUNCTIONS AT THE SOURCE REGION

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Summary

A systematic and novel approach is developed for the dyadic operation of dyadic Green's functions (DGF). The complete forms of the dyadic operation of the DGF for cylindrical waveguides are given. Ambiguities associated with the dyadic operation in the literature are clarified and the errors are redressed.

I. Introduction

The dyadic Green's function (DGF) is a powerful and important tool in electromagnetic theory. The DGF for various geometries (e.g., cylindrical waveguides, spheres, etc.) have been determined by the methods $\bar{\bar{G}}_A$ and $\bar{\bar{G}}_m$. Unfortunately, the property that the dyadic expansions are continuous in some components and discontinuous in the others at the source region has not been recognized yet and there are some errors and ambiguities associated with the dyadic operation of DGF in the literature[1-4].

In this paper, a systematic and novel approach is developed for the dyadic operation of DGF. The derivative of a discontinuous variable is defined in the sense of distribution and performed separately in the components of dyadics. Once the additional terms with δ function which emerge in the generalized derivatives of discontinuous variables are found, we can conveniently use the vector identities to carry out the dyadic operation. As an example, the complete forms of the dyadic operation of the DGF for cylindrical waveguides are given. It is also shown that the electric and the magnetic DGF solved satisfy the dyadic version of Maxwell's equations and of the equation of current continuity.

II. The method of $\bar{\bar{G}}_A$

The vector wave functions for cylindrical waveguides are

$$\bar{L}^{\pm}_{\substack{e\\o}v}(\bar{R})=\nabla f_{\substack{e\\o}v}(\bar{R})=(\nabla\pm\hat{z}jk_g)f_{\substack{e\\o}v}(\rho)\exp(\pm jk_g z)$$

$$=[\bar{l}_{\substack{e\\o}vt}(\rho)\pm\bar{l}_{\substack{e\\o}vz}(\rho)]\exp(\pm jk_g z) \tag{1a}$$

$$\bar{N}^{\pm}_{\substack{e\\o}v}(\bar{R})=\frac{1}{k}\nabla\times\nabla\times[f_{\substack{e\\o}v}(\bar{R})\hat{z}]=\frac{1}{k}[(\pm jk_g\nabla+\hat{z}k_c^2)f_{\substack{e\\o}v}(\rho)]\exp(\pm jk_g z)$$

$$=[\pm\bar{n}_{\substack{e\\o}vt}(\rho)+\bar{n}_{\substack{e\\o}vz}(\rho)]\exp(\pm jk_g z) \tag{1b}$$

$$\bar{M}^{\pm}_{\substack{e\\o}v}(\bar{R})=\nabla\times[f_{\substack{e\\o}v}(\bar{R})\hat{z}]=[\nabla f_{\substack{e\\o}v}(\rho)\times\hat{z}]\exp(\pm jk_g z)$$

$$=\bar{m}_{\substack{e\\o}v}(\rho)\exp(\pm jk_g z) \tag{1c}$$

where f_v is the solution of the scalar Helmholtz equation.
The vector potential DGF $\bar{\bar{G}}_A$ is

$$\bar{\bar{G}}_A(\bar{R}/\bar{R}')=\sum_v\frac{j}{2k_g}[\bar{\bar{S}}_v(\bar{R}/\bar{R}')+\bar{\bar{T}}_v(\bar{R}/\bar{R}')] \tag{2}$$

where

$$\bar{\bar{S}}_v(\bar{R}/\bar{R}')=C_o/k^2[\bar{L}^{+}_{ov}(\bar{R})\bar{L}^{-}_{ov}(\bar{R}')u(z-z')+\bar{L}^{-}_{ov}(\bar{R})\bar{L}^{+}_{ov}(\bar{R}')u(z'-z)]$$

$$=C_o/k^2\{[\bar{l}_{ovt}(\rho)\bar{l}_{ovt}(\rho')+\bar{l}_{ovz}(\rho)\bar{l}_{ovz}(\rho')]\ell_1(z,z')$$

$$+[-\bar{l}_{ovt}(\rho)\bar{l}_{ovz}(\rho')+\bar{l}_{ovz}(\rho)\bar{l}_{ovt}(\rho')]\ell_1'(z,z')\} \tag{3a}$$

$$\bar{\bar{T}}_v(\bar{R}/\bar{R}')=\frac{1}{k_c^2}\{[C_e\bar{M}^{+}_{ev}(\bar{R})\bar{M}^{-}_{ev}(\bar{R}')+C_o\bar{N}^{+}_{ov}(\bar{R})\bar{N}^{-}_{ov}(\bar{R}')]u(z-z')$$

$$+[C_e\bar{M}^{-}_{ev}(\bar{R})\bar{M}^{+}_{ev}(\bar{R}')+C_o\bar{N}^{-}_{ov}(\bar{R})\bar{N}^{+}_{ov}(\bar{R}')]u(z'-z)\}$$

$$=\frac{1}{k_c^2}\{[C_e\bar{m}_{ev}(\rho)\bar{m}_{ev}(\rho')+C_o\bar{n}_{ovt}(\rho)\bar{n}_{ovt}(\rho')+C_o\bar{n}_{ovz}(\rho)\bar{n}_{ovz}(\rho')]\ell_1(z,z')$$

$$+[C_o\bar{n}_{ovt}(\rho)\bar{n}_{ovz}(\rho')-C_o\bar{n}_{ovz}(\rho)\bar{n}_{ovt}(\rho')]\ell_1'(z,z')\} \tag{3b}$$

$$\ell_1(z,z')=\exp[jk_g(z-z')]u(z-z')+\exp[jk_g(z'-z)]u(z'-z) \tag{4a}$$

$$\ell_1'(z,z')=\exp[jk_g(z-z')]u(z-z')-\exp[jk_g(z'-z)]u(z'-z) \tag{4b}$$

The dyadic expansions $\bar{\bar{S}}$ and $\bar{\bar{T}}$ are continuous in the components $\hat{t}\hat{t}$ and $\hat{z}\hat{z}$ and discontinuous in the components $\hat{t}\hat{z}$ and $\hat{z}\hat{t}$ at the source region. With the aid of the generalized derivative

$$\partial\ell_1(z,z')/\partial z=jk_g\ell_1'(z,z') \tag{5a}$$

$$\partial\ell_1'(z,z')/\partial z=jk_g\ell_1(z,z')+2\delta(z-z') \tag{5b}$$

and the vector identities

$$\nabla\times[\varphi(z)\bar{A}]=\varphi(z)\nabla\times\bar{A}+\hat{z}\times[\bar{A}\,\partial\varphi(z)/\partial z] \tag{6}$$

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$$\nabla\nabla\cdot\bar{L}_{ov}(\bar{R}) = -k^2\bar{L}_{ov}(\bar{R}) \tag{7a}$$

$$\nabla\times\bar{M}_{ev}(\bar{R}) = k\bar{N}_{ev}(\bar{R}) \tag{7b}$$

$$\nabla\times\bar{N}_{ov}(\bar{R}) = k\bar{M}_{ov}(\bar{R}) \tag{7c}$$

we obtain

$$\nabla\times\bar{\bar{S}}_v(\bar{R}/\bar{R}') = {C_o}/{k^2}\,\hat{z}\times[-\bar{l}_{ovt}(\rho)\bar{l}_{ovz}(\rho')2\delta(z-z')] \tag{8a}$$

$$\nabla\times\bar{\bar{T}}_v(\bar{R}/\bar{R}') = {C_o}/{k_c^2}\,\hat{z}\times[\bar{n}_{ovt}(\rho)\bar{n}_{ovz}(\rho')2\delta(z-z')] + k\bar{\bar{U}}_v(\bar{R}/\bar{R}') \tag{8b}$$

$$\nabla\cdot\bar{\bar{T}}_v(\bar{R}/\bar{R}') = {C_o}/{k_c^2}\,\hat{z}\cdot[-\bar{n}_{ovz}(\rho)\bar{n}_{ovt}(\rho')2\delta(z-z')] \tag{9a}$$

$$\nabla\nabla\cdot\bar{\bar{T}}_v(\bar{R}/\bar{R}') = {C_o}/{k_c^2}\,\nabla\{\hat{z}\cdot[-\bar{n}_{ovz}(\rho)\bar{n}_{ovt}(\rho')2\delta(z-z')]\} \tag{9b}$$

$$\begin{aligned}\nabla\nabla\cdot\bar{\bar{S}}_v(\bar{R}/\bar{R}') &= {C_o}/{k^2}\,\nabla\{\hat{z}\cdot[\bar{l}_{ovz}(\rho)\bar{l}_{ovt}(\rho')2\delta(z-z')]\}\\ &\quad -k^2\bar{\bar{S}}_v(\bar{R}/\bar{R}') - \hat{z}\hat{z}C_o(-jk_g)f_{ov}(\rho)f_{ov}(\rho')\cdot\\ &\quad 2\delta(z-z')\end{aligned} \tag{9c}$$

where

$$\begin{aligned}\bar{\bar{U}}_v(\bar{R}/\bar{R}') = {1}/{k_c^2}\{&[C_o\bar{M}_{ov}^{+}(\bar{R})\bar{N}_{ov}^{-}(\bar{R}') + C_e\bar{N}_{ev}^{+}(\bar{R})\bar{M}_{ev}^{-}(\bar{R}')]u(z-z')\\ &+[C_o\bar{M}_{ov}^{-}(\bar{R})\bar{N}_{ov}^{+}(\bar{R}') + C_e\bar{N}_{ev}^{-}(R)\bar{M}_{ev}^{+}(\bar{R}')]u(z'-z)\}\end{aligned} \tag{10}$$

Contrary to Kisliuk[1,2], (8) and (9) show the expansions $\bar{\bar{S}}$ and $\bar{\bar{T}}$ are not purely lognitudinal and transverse fields at the source region. With the aid of the relations between $\bar{\bar{G}}_e$, $\bar{\bar{G}}_m$ and $\bar{\bar{G}}_A$ [4], we obtain

$$\bar{\bar{G}}_e(\bar{R}/\bar{R}') = \sum_v[{j}/{2k_g}\,\bar{\bar{T}}_v(\bar{R}/\bar{R}') - \hat{z}\hat{z}\,{C_o}/{k^2}\,f_{ov}(\rho)f_{ov}(\rho')\delta(z-z')] \tag{11a}$$

$$\bar{\bar{G}}_m(\bar{R}/\bar{R}') = \sum_v {j}/{2k_g}\,k\,\bar{\bar{U}}_v(\bar{R}/\bar{R}') \tag{11b}$$

Although the additional terms with δ function which emerge in the generalized derivative of the first-order are cancelled themselves and do not appear in (11), they are very important for us to study and understand the properties of DGF in electromagnetic theory. For example, with the aid of (9a) and the relation

$$\hat{z}\cdot\bar{n}_{ovz}(\rho) = (-{1}/{jk_g})\nabla\cdot\bar{n}_{ovt}(\rho) \tag{12}$$

as well as the completeness relation

$$\bar{\bar{I}}_t\,\delta(\bar{R}-\bar{R}') = \sum_v {1}/{k_c^2}[C_e\bar{m}_{ev}(\rho)\bar{m}_{ev}(\rho') - ({k^2}/{k_g^2})C_o\bar{n}_{ovt}(\rho)\bar{n}_{ovt}(\rho')]\delta(z-z') \tag{13}$$

The following expression can be derived

$$\begin{aligned}\nabla\cdot\sum_v {j}/{2k_g}\,\bar{\bar{T}}_v(\bar{R}/\bar{R}') &= \sum_v {C_o}/{k^2k_c^2}\,\nabla\cdot({k^2}/{k_g^2})\bar{n}_{ovt}(\rho)\bar{n}_{ovt}(\rho')\delta(z-z')\\ &= -{1}/{k^2}\,\nabla\cdot\bar{\bar{I}}_t\,\delta(\bar{R}-\bar{R}')\end{aligned} \tag{14}$$

(14) shows that $\bar{M}$ and $\bar{N}$ are complete for 'transverse' current in cylindrical waveguides even though $\nabla\cdot\bar{J}\neq 0$. It is easy to prove that $\bar{\bar{G}}_e$ and $\bar{\bar{G}}_m$ satisfy the dyadic version of Maxwell's equations and the dyadic version of the equation of current continuity.

III. The method of $\bar{\bar{G}}_m$

If $\bar{\bar{G}}_m$ is first found, $\bar{\bar{G}}_e$ can be derived by means of the dyadic operation defined in this paper. The dyadic operation of DGF was performed in [3,4] by interchanging differential operators and the integral operators in the eigenfunction expansion of DGF. It is important to mention that the separation of the singular terms in the integrations after the interchange of the operators is not unique because the condition $|z-z'|>0$ in Jardan lemma in the complex integral is not satisfied at the source region. A counter-example is easily constructed. It is also shown by (8) and (10) that the interchange of the differential and integral operators fails to reveal the property that $\bar{\bar{S}}$ and $\bar{\bar{T}}$ are no longer purely lognitudinal and transverse fields at the source region, and then fails to determine unique and complete additional terms with δ function. Thus, we conclude that the interchange of the operators is invalid at the source region.

It is worth to point out that Pathak independently obtained the simalar results as (14) by an alternative procedure[5]. In his approach, a condition govening the behavior of the eigenfunction expansion at the source region is needed to derive the final solution for $\nabla\times\bar{\bar{G}}_m$.

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